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SPHERICAL ZONE BOUNDARIES IN PERSPECTIVE PROJECTION

OCTOBER 1966

B. A. Forest

Prepared for

DEPUTY FOR ENGINEERING AND TECHNOLOGY
DIRECTORATE OF COMPUTERS
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

D/C

Project 7070
Prepared by
THE MITRE CORPORATION
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FOREWORD

The author thanks Dr. S. Okada for the interest he has shown in this report and the many useful suggestions he has made.

REVIEW AND APPROVAL

This technical report has been reviewed and is approved.

George E. Vranesh

GEORGE E. VRANESH
1st Lt. USAF
7070 Project Officer

ABSTRACT

This report considers the projection of spherical zone boundaries onto an image plane using the "perspective projection". Special cases of this type of mapping include: the stereographic, the gnomonic, and the orthographic projections.

In addition to proving that the image of any spherical zone boundary is a conic section, the formulation of the parameters of that conic section is given.

An important aspect of the projection technique lies in the capability of having any point of the sphere as the center of projection and any great circle through that point as the centerline of the mapping.

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SECTION I

INTRODUCTION

In the development of analytical or visual aids for aerospace studies there has been a growing need for a technique to produce maps, geographic reference systems, orbital track overlays, etc., quickly and accurately.

This report presents a discussion of mapping called "perspective projection", and its application to the projection of spherical zone boundaries onto a plane.

In Section II basic projection formulas are derived, and the special cases related to well known map projections are given.

In Section III the application to the earth is discussed, and Section IV gives the technique for making any given geographic location the center of projection.

Section V develops the general theory of projecting spherical zone boundaries. In addition to proving that the image is always a conic section (allowing for degenerate cases), the conditions under which certain conic sections arise are given in Section VI.

In Section VII each conic section is discussed in detail. The formulation is given in such a way that programming logic will easily follow.

It is intended that the perspective projection will be set up in a master program to produce maps, geographic reference grids, station reference grids, and various curves for use as overlays, such as orbital tracks, visibility zones, satellite eclipse zones, etc. It is intended that a Benson-Lehner plotter will be used.

Many of the programs mentioned above have been written and are operating. These will be discussed in future working reports.

SECTION II
THE PROJECTION

DESCRIPTION

In the general case one has an object, O, in space; a point of projection, P; and an image plane, I. (See Figure 1.) A line is drawn to a point T on the object (or preimage). If the line PT (extended) intersects the image plane I, the point of intersection T' will be called the image of T.

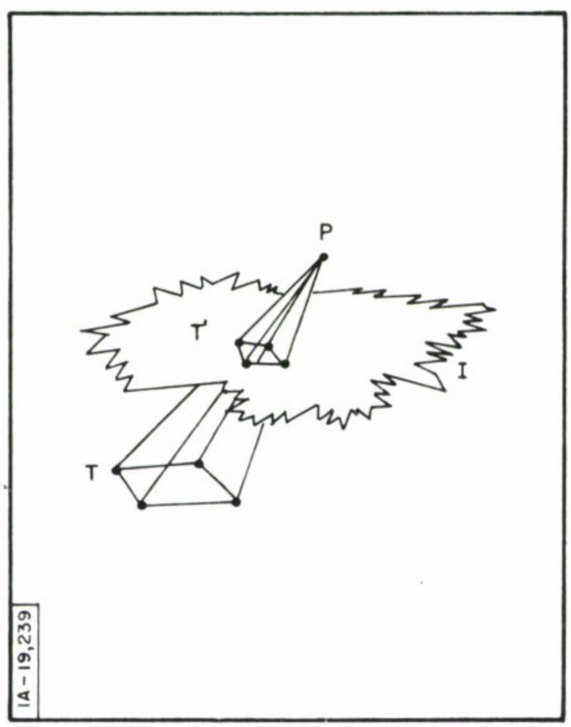

Figure 1. The Projection

The shape of the image depends upon the shape of the object, the orientation of the image plane with respect to the object, and the position of the point of projection.

LIMITATIONS

This discussion is restricted in the following ways:

(a) only those object curves in space are considered which are generated by the intersection of a plane with a given sphere and

(b) the image plane is perpendicular to the line joining the point of projection and the center of the given sphere.

FORMULATIONS

The geometric setup in space is formulated as follows (see Figures 2 and 3).

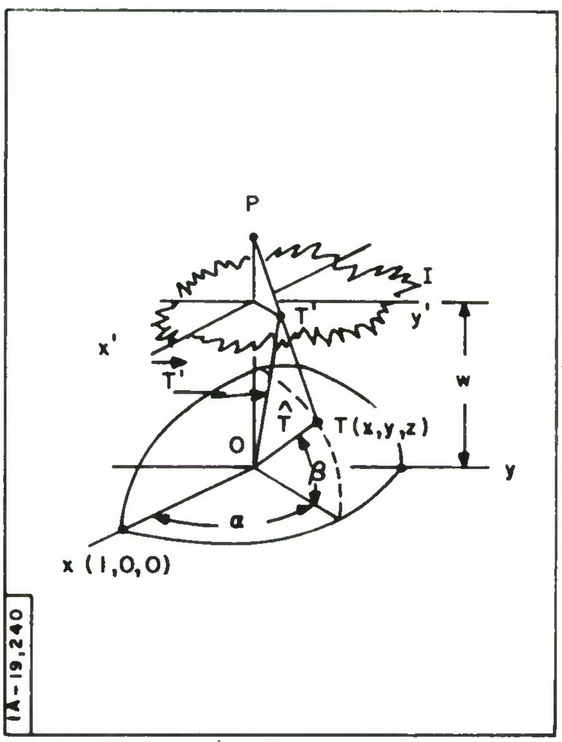

Figure 2. Projection Related to a Unit Sphere

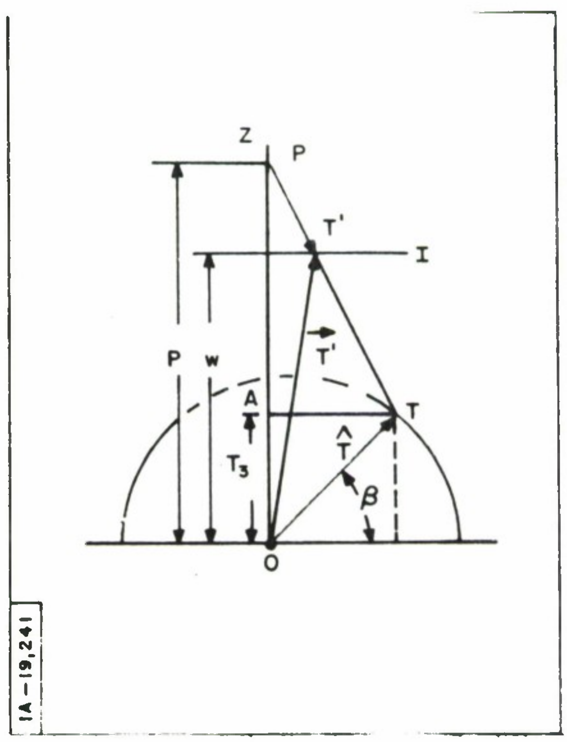

Figure 3. Cross Section of Figure 2

The usual (x, y, z) coordinate system is used with unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ giving the positive direction of the x, y, and z axes respectively. This notation is used with respect to the object. Points of the image will be designated, in the same frame, using coordinates x', y', and z'.

A unit sphere with center at the origin is used whose equation is

$$x^2 + y^2 + z^2 = 1. \tag{1}$$

The point of projection p is given by

$$\vec{p} = p \cdot \hat{k}\,. \tag{2}$$

The image plane is given by

$$z = w. \tag{3}$$

A point T of the object in space has coordinates (x, y, z), and a position vector

$$\hat{T} = x\hat{i} + y\hat{j} + z\hat{k}\,, \tag{4}$$

or, in spherical coordinates (α, β)

$$T = \cos\alpha \, \cos\beta\hat{i} + \sin\alpha \, \cos\beta\hat{j} + \sin\beta\hat{k} \tag{5}$$

Figure 3 shows the geometry resulting from passing a plane through 0, the center of the sphere, the point of projection, p, and the point T to be projected. The position vector of T', the image of T, is given by

$$\vec{T'} = p\hat{k} + \left(\frac{p-w}{p-T_3}\right) [\hat{T} - p\hat{k}] \text{ for } p \neq T_3 \tag{6}$$

where T_1, T_2 and T_3 are the x, y, z components of $\hat{T}$.

Primed letters shall be used to indicate points in the image plane corresponding to points of the sphere.

In the image plane I, we find the coordinates (x', y') of T' as,

$$x' = \vec{T}' \cdot \hat{i} = \frac{p-w}{p-T_3} \cdot T_1$$

$$y' = \vec{T}' \cdot \hat{j} = \frac{p-w}{p-T_3} \cdot T_2 \quad p \neq T_3 \tag{7}$$

Thus any point of the sphere (for which $p \neq T_3$) may be projected onto the image plane from point P by using Equations 7.

PARAMETERS P AND W RELATED TO STANDARD MAP PROJECTIONS

Corresponding to certain values of p and w are standard map projections as defined by cartographers. The choices of p and w and the related map projections are listed in Table I.

Details of a specific projection may be obtained from Reference 1.

Table I

Parameters P and W Related to Standard Map Projections

p	w	Projection	Scale
-1	0	Stereographic	1 unit = 3963 mi.
0	1	Gnomonic	1 unit = 3963 mi.
∞	0	Orthographic	1 unit = 3963 mi.
p	w = 1/p	Geometric projection	1 unit = 3963 mi.

Figure 4 shows the geometry of the projection for each case, using a setup as indicated in FORMULATIONS, beginning on page 4.

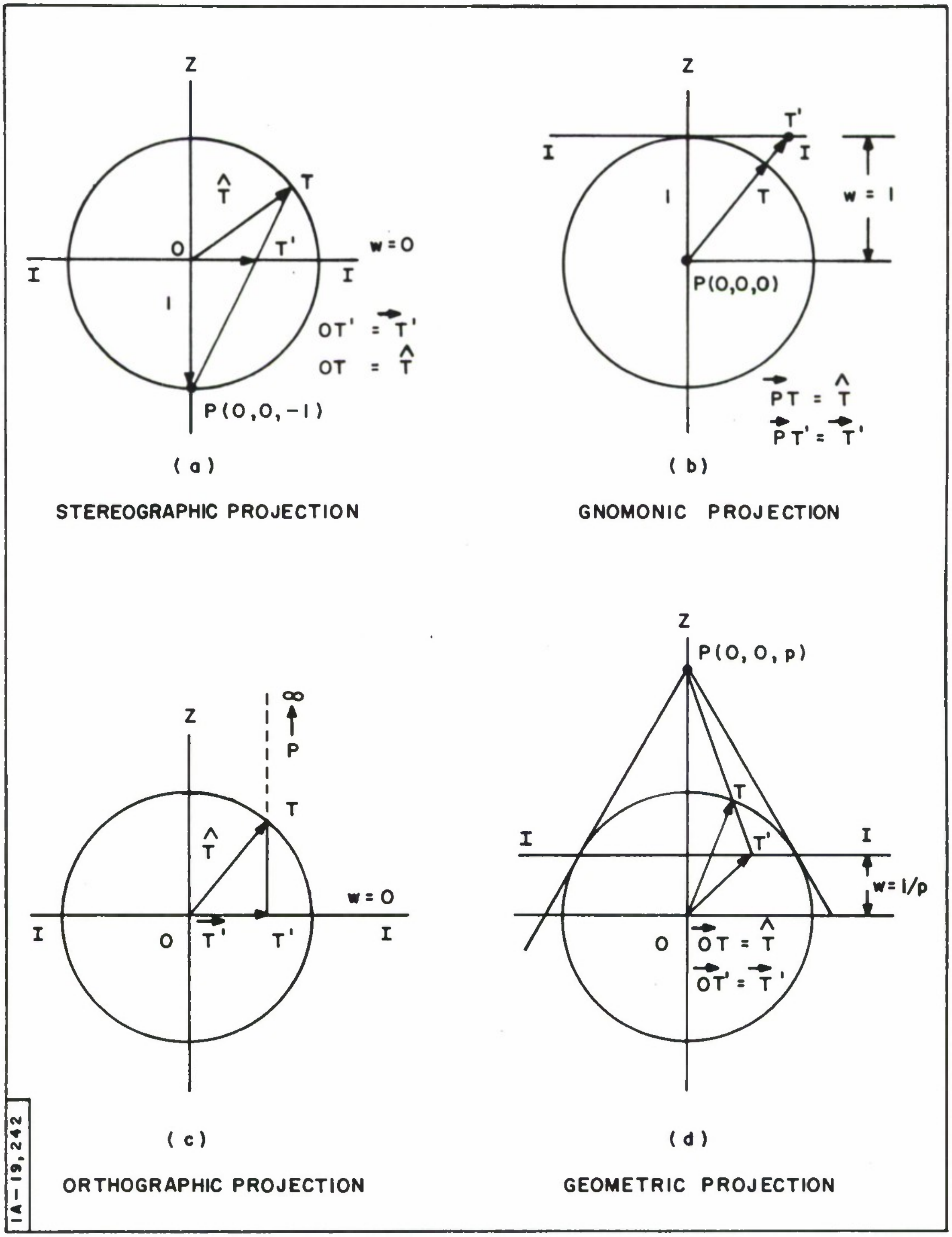

Figure 4. Geometrical Setup for Various Values of P and W

SECTION III

APPLICATION TO THE EARTH

If the unit sphere as given by Equation (1) represents the earth's surface, the following is defined:

(a) the north pole normally has position vector $\hat{k}$ and the south pole $-\hat{k}$.

(b) the Greenwich meridian lies in the x-z plane; the upper meridian of Greenwich is that one-half of the great circle which contains Greenwich.

(c) longitude (east) and latitude will correspond to the usual spherical-coordinate angle measurements.

A point T of the sphere, then, has a position vector,

$$\hat{T} = \cos\phi \; \cos\lambda \hat{i} + \cos\phi \; \sin\lambda \hat{j} + \sin\phi \hat{k}$$

where

λ = east longitude and ϕ = latitude of the position.

Using Equations (7) it is clear that if $p \neq 1$, the north pole projects always to the point (0, 0) in the image plane and the projection is referred to as the "normal" projection.

SECTION IV

CHANGING THE CENTER OF PROJECTION

The foregoing formulation restricts one to a map in which the North Pole is the center of projection and the Greenwich meridian is the centerline of the map. It is desirable to have other geographic locations as the center of the projection and also to have any great circle through this point as the centerline of the map. This section sets forth the theory and formulation necessary to accomplish these results.

As depicted in Figure 5, the new center of projection is chosen as the point $C(\lambda_C, \phi_C)$ where (λ_C, ϕ_C) are respectively the east longitude and latitude of C. A certain great circle through C is determined by choosing a

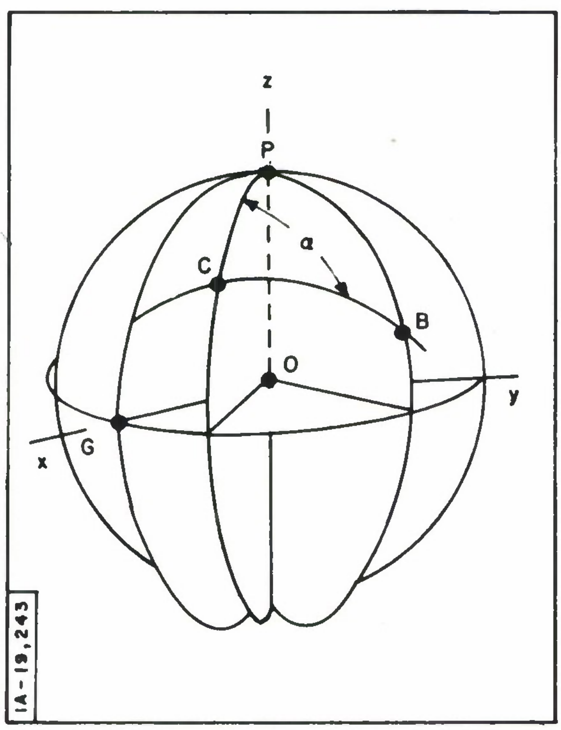

Figure 5. Center of Projection

second point $B(\lambda_B, \phi_B)$ not at C. The great circle containing arc $\widehat{CB}$ makes an angle, α, with the meridian of C. Note here that α is the azimuthal angle of B measured clockwise from the meridian of C. Clearly $0 \leq \alpha \leq 2\pi$. Finally, the Greenwich meridian contains arc $\widehat{GP}$, and the coordinate axes x and z contain respectively, the segments OG and OP with O as the center of the sphere. The axis of y makes a right-handed system with the x and z axes.

A transformation R is now developed which will do the following, in order.

It will rotate the sphere about the z-axis so that point C (or the meridian of C) moves through the angle, $-\lambda_C$; C then lies on the prime meridian. This first rotation is given by the matrix,

$$D = \begin{pmatrix} \cos\lambda_C & \sin\lambda_C & 0 \\ -\sin\lambda_C & \cos\lambda_C & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It will then rotate the sphere about the y-axis through the angle, $-(\pi/2 - \phi_C)$, bringing the point C to the position, (0, 0, 1), formerly occupied by the North Pole. This second rotation is given by the matrix,

$$C = \begin{pmatrix} \cos(\pi/2 - \phi_C) & 0 & -\sin(\pi/2 - C) \\ 0 & 1 & 0 \\ \sin(\pi/2 - C) & 0 & \cos(\pi/2 - C) \end{pmatrix} = \begin{pmatrix} \sin\phi_C & 0 & -\cos\phi_C \\ 0 & 1 & 0 \\ \cos\phi_C & 0 & \sin\phi_C \end{pmatrix}.$$

Finally, there is a rotation of the sphere about the z-axis through the angle $(\pi + \alpha)$ carrying the point B counterclockwise to a position on the prime meridian. This last rotation is given by the matrix,

$$B = \begin{pmatrix} \cos(\pi + \alpha) & -\sin(\pi + \alpha) & 0 \\ \sin(\pi + \alpha) & \cos(\pi + \alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\alpha & +\sin\alpha & 0 \\ -\sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The angle α is found as follows.

Let $\hat{C}$ and $\hat{B}$ be the position vectors of points C and B respectively and $\hat{P} = \hat{K}$ be the position vector of the pole, then

$$\cos\alpha = \frac{(\hat{C} \times \hat{P}) \cdot (\hat{C} \times \hat{B})}{\cos\phi_C \cdot \sin\delta}$$

$$= \frac{\cos\phi_C \sin\phi_B - \cos\phi_B \sin\phi_C \cos\lambda_C - \lambda_B}{\sin\delta}$$

where $\delta = \cos^{-1}(\hat{C} \cdot \hat{B})$ is the distance from C to B in radians. This formula makes use of the fact that the azimuth angle measures the dihedral angle between the planes containing arcs $\overset{\frown}{CB}$ and $\overset{\frown}{CP}$. This dihedral angle is the angle between vectors drawn perpendicular to the planes. (One must be careful of the direction of these vectors.)

From the spherical triangle CPB we get the relationship

$$\sin\alpha = \frac{\sin(\lambda_B - \lambda_C)\cos\phi_B}{\sin\delta} \quad . \tag{10}$$

There is now sufficient information to assign quadrants.

The product matrix $R = BCD$ follows as:

$$R = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

and in which

$$\begin{aligned}
a_{11} &= -\left(\cos \alpha \sin \phi_C \cos \lambda_C + \sin \alpha \sin \lambda_C\right), \\
a_{12} &= -\cos \alpha \sin \phi_C \cos \lambda_C + \sin \alpha \cos \lambda_C, \\
a_{13} &= \cos \alpha \cos \phi_C, \\
a_{21} &= -\sin \alpha \sin \phi_C \cos \lambda_C + \cos \alpha \sin \lambda_C, \\
a_{22} &= -\left(\sin \alpha \sin \phi_C \sin \lambda_C + \cos \alpha \cos \lambda_C\right), \\
a_{23} &= \sin \alpha \cos \phi_C, \\
a_{31} &= \cos \phi_C \cos \lambda_C, \\
a_{32} &= \cos \phi_C \sin \lambda_C, \quad \text{and} \\
a_{33} &= \sin \phi_C \,.
\end{aligned} \tag{11}$$

Thus, any position vector $\hat{A}_0$ of a given point A of the sphere will be rotated into a new position vector

$$\hat{A} = R\hat{A}_0.$$

The position vector $\hat{A}_0\left(\lambda_0, \phi_0\right)$ before rotation is given by

$$\hat{A}_0 = \cos \phi_0 \cos \lambda_0 \hat{i} + \cos \phi_0 \sin \lambda_0 \hat{j} + \sin \phi_0 \hat{k}.$$

Applying the transformation given by Equations (11) the components (a, b, c) of $\hat{A}$ are derived as:

$$a = a_{11} \cos \phi_0 \cos \lambda_0 + a_{12} \cos \phi_0 \sin \lambda_0 + a_{13} \sin \phi_0,$$

$$b = a_{21} \cos \phi_0 \cos \lambda_0 + a_{22} \cos \phi_0 \sin \lambda_0 + a_{23} \sin \phi_0, \text{ and}$$

$$c = a_{31} \cos \phi_0 \cos \lambda_0 + a_{32} \cos \phi_0 \sin \lambda_0 + a_{33} \sin \phi_0.$$

Finally, on applying Equations 7

$$x' = \left(\frac{p - w}{p - c}\right) a$$

$$y' = \left(\frac{p - w}{p - c}\right) b \qquad \text{for } p \neq c$$

The point $(x' \ y')$ is the image of the point A.

SECTION V

PROJECTING SPHERICAL ZONE BOUNDARIES

In this section the projection of spherical zone boundaries from the sphere onto the image plane is considered. A spherical zone boundary is a circle in space and is the intersection of a plane with the sphere. The plane is uniquely determined by a unit vector

$$\hat{A} = a\hat{i} + b\hat{j} + c\hat{k}$$

normal to the plane and a fixed point B in the plane whose position vector is $\vec{B}$.

If $\vec{x}$ is the position vector of any point (x, y, z) in the plane, then

$$(\vec{x} - \vec{B}) \cdot \hat{A} = 0$$

defines the plane (see Figure 6).

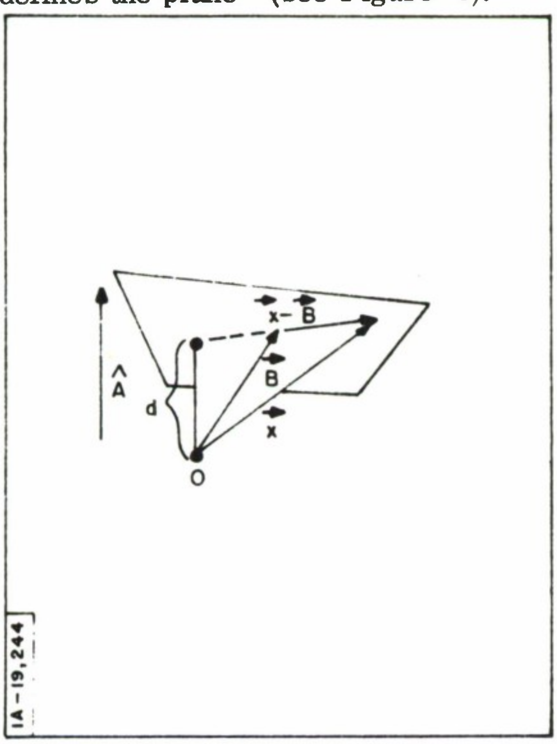

Figure 6. Geometry of Defining the Plane

Equation (12) reduces to

$$\hat{A} \cdot \vec{x} = \vec{B} \cdot \hat{A} = d$$

where d is the directed distance from the origin to the plane.

It is required that $-1 \leq d \leq 1$. If $d^2 = 1$, then the plane is tangent to the sphere and we have a null circle. If $d = 0$, the plane contains the center of the sphere and the zone boundary is a great circle. In fact, the zone boundary is a great circle if and only if $d = 0$.

Assume that the point to be projected lies on the plane given by Equation (12) viz:

$$ax + by + cz = \hat{A} \cdot \vec{B} = d \tag{13}$$

This plane will be referred to as the generating plane, G.

Now, Equations (7) relate a point $T(x, y, z)$ in space to a point $T'(x' y')$ in the image plane. If the point T lies in the plane given by Equation (13),

$$x = \frac{(pc - d) \cdot x'}{[c(p - w) - (ax' + by')]} ,$$

$$y = \frac{(pc - d)\ y'}{[c(p - w) - (ax' + by')]} , \quad \text{and}$$

$$z = \frac{d(p - w) - p(ax' + by')}{c(p - w) - (ax' + by')} . \tag{14}$$

If the point $T(x, y, z)$ is further constrained to lie on the sphere given by Equation (1), the following condition exists,

$$x'^2\left[(d-pc)^2 + a^2\left(p^2-1\right)\right] + y'^2\left[(pc-d)^2 + b^2\left(p^2-1\right)\right]$$

$$+ 2ab\left(p^2-1\right) x' y' + 2(p-w)(c-pd)(ax' + by')$$

$$+ (p-w)^2\left(d^2-c^2\right) = 0 \tag{15}$$

Equation (15) clearly defines the image to be a conic section of the form

$$Au^2 + Buv + Cv^2 + Du + Ev + F = 0. \tag{16}$$

By using the methods set forth in Reference 2, page 256, and the notation of Equation (16), we find the following equation:

$$B^2 - 4AC = -4\left[(d - pc)^2\right]\left[(d - pc)^2 + \left(a^2 + b^2\right)\left(p^2 - 1\right)\right] \tag{17}$$

and the discriminant (see the Appendix).

$$\Delta = \frac{1}{2}\begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = -4(p-w)^2 (d-pc)^4 \left(1 - d^2\right). \tag{18}$$

The type of conic that one gets for an image is summarized in Table II.

Table II

Type of Conic Sections Related to Values of $B^2 - 4AC$ and Δ

	$B^2 - 4AC < 0$	$B^2 - 4AC = 0$	$B^2 - 4AC > 0$
$\Delta \neq 0$	Ellipse if $A\Delta < 0$ No locus if $A\Delta > 0$	Parabola	Hyperbola
$\Delta = 0$	Point (null ellipse)	2 parallel lines, a single line, or no locus	2 intersecting lines.

Now analyze Equations (17) and (18) to determine the geometric situation for which $\Delta = 0$ and $B^2 - 4AC$ is positive, negative, or zero. For this and later applications, it is convenient to write Equation (15) in a different form, viz.:

$$(d - pc)^2 \left(x'^2 + y'^2\right) + \left[(p + 1)(ax' + by') - (d + c)(p - w)\right]$$
$$\left[(p - 1)(ax' + by') - (d - c)(p - w)\right] = 0. \qquad (19)$$

To simplify the analysis and the geometry, it will be helpful to rotate the sphere about the z-axis in such a way that the vector A goes into a vector A' which lies in the x-z plane and $\hat{A}' \cdot \hat{i} \geq 0$.

Using the matrix

$$R = \begin{pmatrix} \frac{a}{m} & \frac{b}{m} & 0 \\ -\frac{b}{m} & \frac{a}{m} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad m = \sqrt{a^2 + b^2}, \qquad (20)$$

we have

$$\hat{A}' = R\hat{A} = m\hat{i} + c\hat{k} \quad . \tag{21}$$

Consider first, the case $\hat{A}' \cdot i \neq 0$, that is the generating plane is not parallel to the x-y plane.

Figure 7 shows the trace in the x-z plane. The line GR is the trace of the generating plane, G. QQ' is the projection of that spherical zone boundary generated by the intersection of plane G with the sphere. I is the trace of the image plane, and p is the point of projection.

We shall have use for a unit vector $\hat{N}$ such that $\hat{N} \cdot \hat{A} = 0$; $\hat{N} \cdot \hat{j} = 0$; the desired vector is

$$\hat{N}' = - c\hat{i} + m\hat{k}$$

in the rotated system, and

$$\hat{N} = \frac{-ac}{m}\hat{i}\,\frac{bc}{m}\hat{j} + m\hat{k} \tag{22}$$

in the nonrotated system. Clearly the vector $\hat{N}'$ is normal to $\hat{A}'$, lies in the x-z plane, and has a positive z-component $(m > 0)$.

It is clear that the points Q and Q' are the points on the circular zone boundary which have the maximum and minimum z coordinates respectively. (Note that since $\hat{A}'$ is normal to plane G, the rotation brings the plane, G, perpendicular to the x-z plane, hence the trace GR is also the projection of plane G on the x-z plane.)

The position vectors of Q and Q' are,

$$\hat{0Q} = \hat{Q} = d\hat{A}' + \sqrt{1 - d^2}\,\hat{N}'$$

and

$$\hat{0Q}' = \hat{Q}' = d\hat{A}' - \sqrt{1 - d^2}\,\hat{N}' \quad . \tag{23}$$

Since $d^2 \leq 1$, Δ is never positive. Consider now each of the factors in the right member of Equation (18):

$$\text{(a)} \quad p - w = 0 \text{ if } p = w;$$

that is, if the image plane contains the point of projection. In a practical situation $p \neq w$;

$$\text{(b)} \quad 1 - d^2 = 0 \text{ if } d = \pm 1;$$

that is, if the generating plane is tangent to the sphere. In this case the degeneracy arises from the fact that a point is projected. In particular, the point is either that whose position vector is $\hat{A}$ for $d = +1$ or the antipode of this point for $d = -1$.

$$\text{(c)} \quad d - pc = 0$$

if the generating plane contains the point of projection. This is shown as follows (see Figure 7). The equation of the line RG is

$$mx + cz = d \ . \tag{24}$$

The z-intercept of this line is, in general, the point at which the generating plane cuts the z-axis. Thus (see Figure 7).

$$\overrightarrow{OR} = \frac{d}{c} \hat{k} \text{ for } c \neq 0 \ .$$

If the generating plane contains the point P, we have $d/c = p$ or

$$d - pc = 0 \ . \tag{25}$$

Thus $d - pc = 0$ if and only if the generating plane contains the point of projection.

If $c = 0$, the generating plane is parallel to the z-axis and from Equation (25) we infer that it is necessary that $d = 0$ in order for the

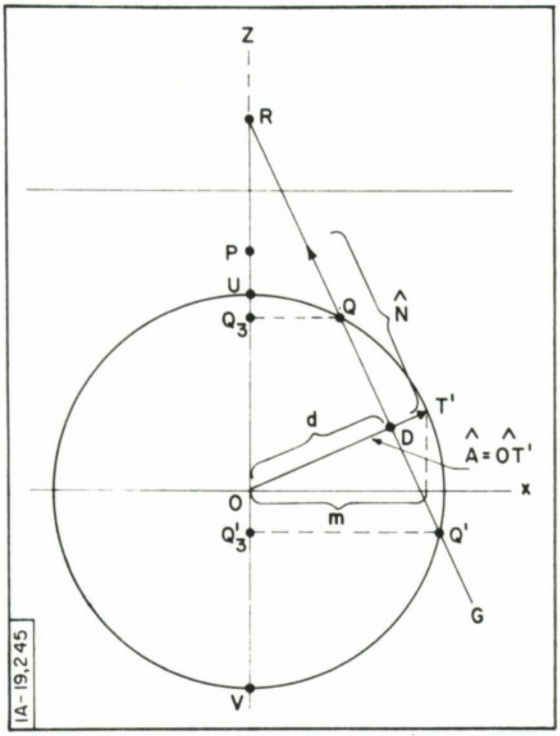

Figure 7. Trace in X-Z Plane

generating plane to contain the point of projection. In summary, a degenerate case is always present when a great circle passes through the center of projection.

Consider now the factors of the right member of Equation (17):

(a) $(d - pc)^2 \geq 0$. We have considered this before as to the equality with zero. Since this factor is never negative, it will not affect the sign of $(B^2 - 4AC)$.

(b) $F = \left[(d - pc)^2 + \left(a^2 + b^2\right)\left(p^2 - 1\right)\right]$. This factor is not so easily disposed of. If $(d - pc) \neq 0$, then $B^2 - 4AC$ is positive, negative, or zero, according as F is negative, positive, or zero, respectively.

If $p^2 \geq 1$, then $F \geq 0$. Thus only those values of p for which the point of projection lies inside the sphere, i.e., $p^2 < 1$ need be considered.

Project the line segment QQ' (see Figure 7) onto the z-axis. The end points of this projection are Q_3 and Q_3', the Z components respectively of $\hat{Q}$ and $\hat{Q}'$, Q_3 and Q_3' form a trichotomy of the diameter UV. Consider the following cases:

(1) $Q_3' < Q_3 < p$,

(2) $Q_3' < Q_3 = p$,

(3) $Q_3' < p < Q_3$,

(4) $Q_3' = p < Q_3$, and

(5) $p < Q_3' < Q_3$.

We find, using Equations (23),

$$Q_3 = \hat{Q} \cdot \hat{k} = dc + m\sqrt{1 - d^2}$$

and

$$Q_3' = \hat{Q}' \cdot \hat{k} = dc - m\sqrt{1 - d^2}\,. \tag{27}$$

If $Q_3' \leq p < Q_3$, we have

$$-m\sqrt{1 - d^2} + dc \leq p \leq m\sqrt{1 - d^2} + dc$$

then

$$-m\sqrt{1 - d^2} \leq p - dc \leq m\sqrt{1 - d^2}$$

or

$$(p - dc)^2 \leq m^2\left(1 - d^2\right)$$

then

$$(p - dc)^2 - m^2\left(1 - d^2\right) \leq 0 . \tag{28}$$

By expanding and rearranging Equation (28) is equivalent to

$$F = (d - pc)^2 - m^2\left(1 - p^2\right) \leq 0 . \tag{29}$$

Since each of the above steps is reversible, we have proved the theorem: If $(p - dc) \neq 0$ and $p^2 < 1$, then $B^2 - 4AC \geq 0$ if and only if $Q_3' \leq p \leq Q_3$. If the equality holds, then $B^2 - 4AC = 0$.

We have taken care of Cases 2, 3, and 4. We now consider Cases 1 and 5.

Factoring Equation (28) we have

$$F = \left(p - dc + m\sqrt{1 - d^2}\right)\left(p - dc - m\sqrt{1 - d^2}\right) \tag{30}$$

and from Equations (27)

$$F = \left(p - Q_3'\right)\left(p - Q_3\right). \tag{31}$$

Now $F > 0$ if both factors of the right member have the same sign, i.e.,

$$p > Q_3' \text{ and } p > Q_3 \tag{32}$$

or

$$p < Q_3' \text{ and } p < Q_3 . \tag{33}$$

Inequalities (32) give $Q_3' < Q_3 < p$, and inequalities (33) give $p < Q_3' < Q_3$ and we have the theorem:

If $(p - dc) \neq 0$, then $B^2 - 4AC < 0$, if and only if $p > Q_3$ or $p < Q_3'$.

If $B^2 - 4AC < 0$ and $\Delta \neq 0$, the locus is an ellipse — if, in addition, $A\Delta$ (or $C\Delta$) < 0. There is no locus if $A\Delta$ (or $C\Delta$) > 0.

Since $\Delta \leq 0$ [see Equation (18)], it remains only to show that $A > 0$, (or $C > 0$).

It has already been shown that

$$B^2 - 4AC < 0 \text{ if } (d - pc)^2 + \left(a^2 + b^2\right)\left(p^2 - 1\right) > 0 . \tag{34}$$

Equation (34) leads to

$$\left(p^2 - 1\right) > - \frac{(d - pc)^2}{\left(a^2 + b^2\right)} ,$$

then multiplying by A_i^2,

$$A_i^2\left(p^2 - 1\right) > \frac{- A_i^2 (d - pc)^2}{\left(a^2 + b^2\right)} \qquad \begin{array}{l} A_i = a,\ b \\ A_i \neq 0 \text{ for some } i \end{array}$$

and finally,

$$(d - pc)^2 + A_i^2\left(p^2 - 1\right) > (d - pc)^2 \left(\frac{a^2 + b^2 - A_i^2}{a^2 + b^2}\right) > 0 \qquad \begin{array}{l} A_i = a,\ b \\ A_i \neq 0 \text{ for some } i . \end{array} \tag{35}$$

Now,

$$(d - pc)^2 + A_i^2\left(p^2 - 1\right) = A \quad \text{for } A_i = a$$
$$= C \quad \text{for } A_i = b$$

and by Equation (35) $A > 0$

and $C > 0$ when $B^2 - 4AC < 0$

Hence $A\Delta < 0$ and $C\Delta < 0$ and the locus is an ellipse.

Now consider the case $|c| = 1$. If $c = \pm 1$, then $a = b = 0$.
Equation (19) becomes

$$(d - pc)^2\left(x'^2 + y'^2\right) + [(d \pm 1)(p - w)]\,[(d \mp 1)(p - w)] = 0$$

$$(d - pc)^2\left(x'^2 + y'^2\right) = \left(1 - d^2\right)(p - w)^2$$

from Equation 17

$$B^2 - 4AC = -4(d - pc)^4$$

and

$$\Delta = -4(p - w)^2(d - pc)^4\left(1 - d^2\right).$$

If $(d - pc) = 0$, there is no locus.

If $(d - pc) \neq 0$, the image is a circle with center $(0, 0)$ and radius

$$R = \sqrt{\frac{\left(1 - d^2\right)(p - w)^2}{(d - pc)^2}} = \left|\frac{p - w}{(d - pc)}\right| \sqrt{1 - d^2}\,.$$

The circle is a null circle if $p = w$ or if $d^2 = 1$.

SECTION VI

APPLICATIONS

In order to illustrate the above theory, the projection of a given spherical zone boundary will be shown for various values of p and w.

To define the zone boundary, take

$$\hat{A}_0 = \frac{1}{2}(\hat{i} + \hat{j} + \sqrt{2}\,\hat{k}) \text{ and } d = \frac{\sqrt{2}}{2}.$$

A further requirement is that the 315° meridian lie along the positive x axis and that the north pole be the center of projection. Thus

$$\hat{C} = \hat{k} \text{ and } \hat{B} = (\cos 315^{o})\hat{i} + (\sin 315^{o})\hat{j} = \frac{\sqrt{2}}{2}(\hat{i} - \hat{j})$$

From Equation (10) we get

$$\alpha = 225^{o}$$

and from Equation (11)

$$R = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (37)$$

Then,

$$\hat{A} = R \cdot \hat{A}_0 = 0 \cdot \hat{i} + \frac{\sqrt{2}}{2}\hat{j} + \frac{\sqrt{2}}{2}\hat{k} \quad (38)$$

The geometry is shown in Figure 8.

Consider five cases:

(1) $p = -1;\ w = 0,$

(2) $p = 0;\ w = 1,$

(3) $p = \frac{1}{2};\ w = 1,$

(4) $p = 1;\ w = 0,$ and

(5) $p = 2;\ w = 0$

<u>Case 1</u>. $p = -1;\ w = 0$

This is a stereographic projection. Equation (19) reduces to

$$x'^2 + y'^2 - y' = 0$$

or

$$x'^2 + \left(y' - \frac{1}{2}\right)^2 = \frac{1}{4} .$$

Thus the image is a circle with center at $\left(0, \frac{1}{2}\right)$ and radius $\frac{1}{2}$.

With

$$\hat{A}' = \frac{\sqrt{2}}{2}\,\hat{j} + \frac{\sqrt{2}}{2}\,\hat{k} ,$$

$$\hat{N}' = \frac{-\sqrt{2}}{2}\,\hat{j} + \frac{\sqrt{2}}{2}\,\hat{k} \quad \text{(see Equation 22)},$$

$$\hat{Q} = \frac{\sqrt{2}}{2}\,\hat{A}' + \frac{\sqrt{2}}{2}\,\hat{N}' = \hat{k} ,$$

and finally

$$\hat{Q}' = \hat{j} ;$$

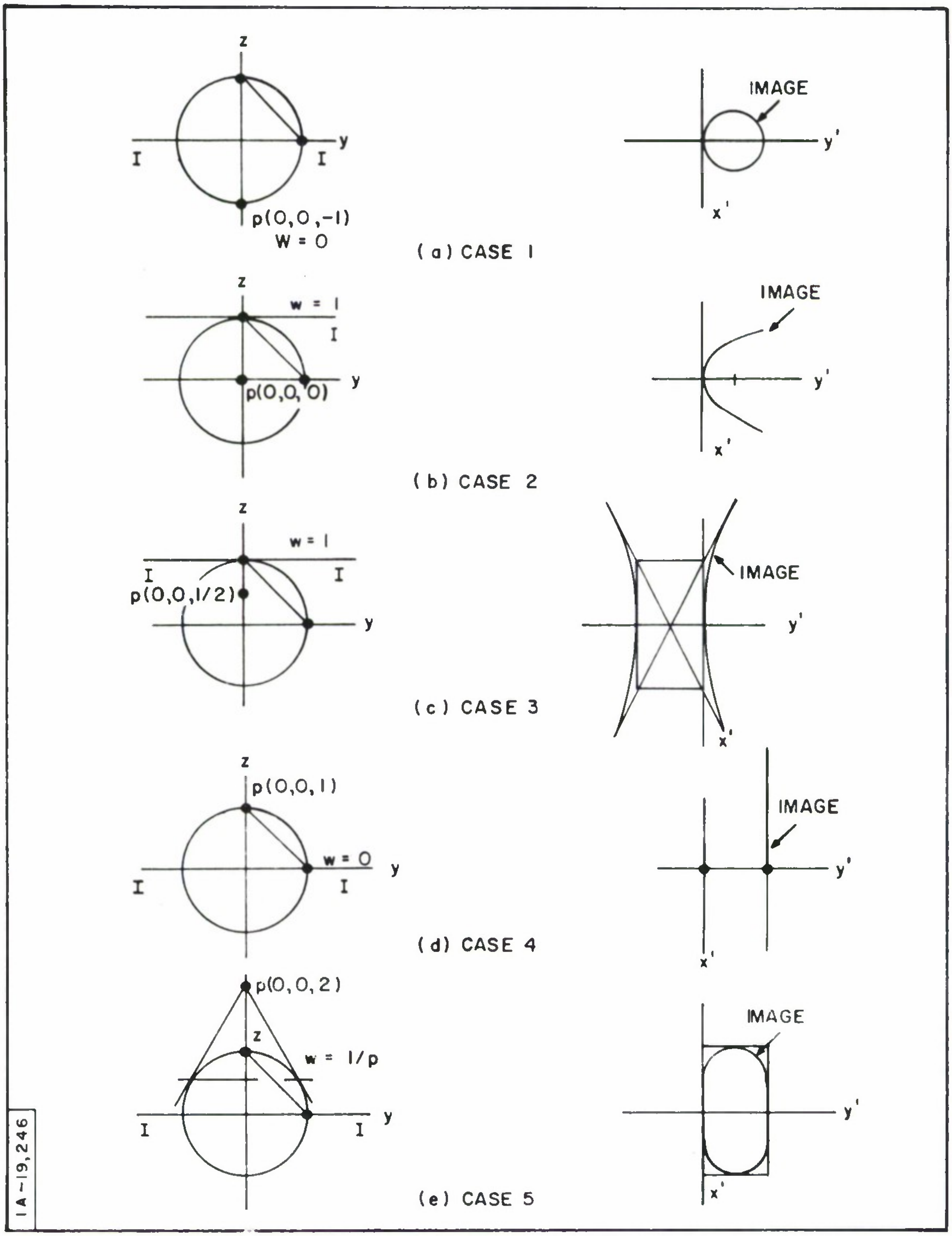

Figure 8. Projection of Spherical Zone Boundary for Various Values of P and W

then $Q_3 = 1$ and $Q_3{}' = 0$. Since $P = -1 < Q_3{}'$, the result is as expected from Equation (34) and Table II.

Case 2. $p = 0$, $w = 1$

This is the gnomonic projection. Using a similar procedure as in Case 1 we find, since $p = Q_3{}'$, that the image is the parabola

$$x'^2 = 2y' \ .$$

Case 3. $p = \frac{1}{2}$, $w = 1$.

This is a "geometric" projection. Since $Q_3{}' < p < Q_3$, we find the image to be the hyperbola

$$x'^2 - 2y'^2 - 2y' = 0$$

or

$$-\frac{x'^2}{\frac{1}{2}} + \frac{\left(y' + \frac{1}{2}\right)^2}{\frac{1}{4}} = 1 \ .$$

Case 4. $p = 1$, $w = 0$

In this case, $p = Q_3$ and, since the generating plane contains the point of projection, $p - dc = 0$. Using Equation (19), the image is the line $y' = 1$.

Case 5. $p = 2$, $w = 0$

Here, $p > Q_3$ and the image is the ellipse

$$\frac{x'^2}{1} + \frac{\left(y' - \frac{1}{2}\right)^2}{\frac{1}{4}} = 1 \ .$$

SECTION VII

DISCUSSION OF THE CONIC SECTIONS

THE NONDEGENERATE ELLIPSE, HYPERBOLA, AND PARABOLA

For this case $\Delta \neq 0$ and $B^2 - 4AC \neq 0$.

Simplify Equation (19) by using the rotation given in Equation (20). This rotation removes the $x'y'$ and y' terms. The resulting equation is

$$x'^2\left[(d - pc)^2 + m^2\left(p^2 - 1\right)\right] + y'^2(d - pc)^2 - 2mx'(p - w)(pd - c)$$

$$+ \left(d^2 - c^2\right)(p - w)^2 = 0 . \tag{39}$$

Noting that

$$(d - pc)^2 + m^2\left(p^2 - 1\right) \equiv (pd - c)^2 + \left(1 - d^2\right)\left(p^2 - 1\right),$$

put Equation (39) in a standard form,

$$\frac{\left[x' - \dfrac{m(p - w)(pd - c)}{(pd - c)^2 + \left(1 - d^2\right)\left(p^2 - 1\right)}\right]^2}{\dfrac{(p - w)^2\left(1 - d^2\right)(d - pc)^2}{\left[(pd - c)^2 + \left(1 - d^2\right)\left(p^2 - 1\right)\right]^2}} + \frac{y'^2}{\dfrac{(P - w)^2\left(1 - d^2\right)}{(pd - c)^2 + \left(1 - d^2\right)\left(p^2 - 1\right)}} = 1 . \tag{40}$$

Now, with

$$F = (pd - c)^2 + \left(1 - d^2\right)\left(p^2 - 1\right) \equiv (d - pc)^2 + m^2\left(p^2 - 1\right)$$

$$\equiv (p - dc)^2 + m^2\left(d^2 - 1\right) ,$$

the center of the conic is $H(x_0', y_0')$ where,

$$x_0' = \frac{m(p - w)(pd - c)}{F}$$

and

$$y_0' = 0 \ .$$

The axes of the conic sections given by Equation (40) are such that one axis is always on a radial line from the center of projection. This is referred to as the "radial axis," and the length of the semiradial axis is

$$r = \sqrt{\frac{(p - w)^2 (d - pc)^2 \left(1 - d^2\right)}{F^2}} \ . \tag{41}$$

The axis normal to r is the "transverse" axis, and the length of the semi-transverse axis is

$$t = \sqrt{\frac{(p - w)^2 \left(1 - d^2\right)}{|F|}} \tag{42}$$

Note here, that when $F < 0$, the conic is a hyperbola, and for $F > 0$, the conic is an ellipse.

Using the inverse rotation to Equation (20),rotate the system to its original position. The position vector of the center of the conic thus is

$$\vec{H} = R^{-1} \begin{pmatrix} x_0' \\ y_0' \\ w \end{pmatrix} = \frac{a(p - w)(pd - c)}{F} \hat{i} + \frac{b(p - w)(pd - c)}{F} \hat{j} + w\hat{k} \tag{43}$$

and, in the image plane, the cartesian coordinates of the center are

$$x_H' = \frac{a(p - w)(pd - c)}{F}$$

and

$$y_H' = \frac{b(p - w)(pd - c)}{F} . \tag{44}$$

Since c is invariant under the rotation used, the semiradial and semitransverse axes are as given by Equations (41) and (42), respectively.

F < 0

If $F < 0$, the conic is a hyperbola.

The coordinates of the foci f_1 and f_2 are

$$f_{1,x} = x_H' + \frac{ga}{m}$$

$$f_{1,y} = y_H' + \frac{gb}{m} ; \tag{45}$$

and

$$f_{2,x} = x_H' - \frac{ga}{m}$$

$$f_{2,y} = y_H' - \frac{gb}{m} \quad \text{for } m \neq 0 ; \tag{46}$$

and

$$g = \sqrt{\frac{(p - w)^2\left(1 - d^2\right)(d - pc)^2}{F^2} - \frac{(p - w)^2\left(1 - d^2\right)}{F}}$$

$$= \left|\frac{p - w}{F}\right| \left(1 - d^2\right) \sqrt{1 - p^2} ; \quad \text{note: } -1 < p < +1 . \tag{47}$$

$\underline{F > 0}$

If $F > 0$, the conic is an ellipse.

If $r > t$, the coordinates of the foci f_1 and f_2 are the same as given by Equations (45) and (46) except that

$$g = \sqrt{\left| \frac{(p - w)^2 \left(1 - d^2\right)(d - pc)^2}{F^2} - \frac{(p - w)^2 \left(1 - d^2\right)}{F} \right|} \tag{48}$$

If $r < t$, the foci are on the transverse axis. To locate these foci consider the ellipse as given in Equation (40).

The position vector of the center, H, is

$$\vec{H}' = x_0{}' \, \hat{i} + w\hat{k} \; .$$

If $x_0{}' > 0$, the focus which is angularly displaced from the radial axis in a positive direction has the position vector

$$\vec{f}'_{1,\,+} = x_0{}' \, \hat{i} + g\hat{j} + w\hat{k} \; ,$$

and the other focus is given by

$$\vec{f}'_{2,\,-} = x_0{}' \, \hat{i} - g\hat{j} + w\hat{k} \; .$$

If $x_0{}' < 0$, the direction of the angular displacements are reversed.

Using the inverse rotation as in Equation (20), rotate the system to its original position and find

$$\vec{f}_{1,\,+} = \frac{1}{m} \left[\left(ax_0{}' - bg\right) \hat{i} + \left(bx_0{}' + ag\right) \hat{j}\right]$$

and

$$\vec{f}_{2,} \; - = \frac{1}{m}\left[\left(ax_0{}' + bg\right)\hat{i} + \left(bx_0{}' - ag\right)\hat{j}\right] \tag{50}$$

Since $r > 0$, and $t > 0$, to determine the conditions for which $r < t$, let

$$r^2 < t^2$$

or

$$\frac{(p - w)^2\left(1 - d^2\right)(d - pc)^2}{F^2} < \frac{(p - w)^2\left(1 - d^2\right)}{F};$$

then

$$(d - pc)^2 < F = (d - pc)^2 + m^2\left(p^2 - 1\right)$$

and we have

$$m^2\left(p^2 - 1\right) > 0 . \tag{51}$$

Thus for $m \neq 0$, $p^2 > 1$ is the condition for which the transverse axis is the longer axis.

If $r = t$, the conic is a circle. The conditions for which $r = t$ follow from Equation (51);

$$m^2\left(p^2 - 1\right) = \left(1 - c^2\right)\left(p^2 - 1\right) = 0 ,$$

gives

$$c = \pm 1$$

and

$$p = \pm 1 \tag{52}$$

as the required conditions.

Thus a circle is possible only in the stereographic projection $(p = \pm 1)$ or if $\hat{A} = \hat{k}$, i.e., if the generating plane is perpendicular to the z-axis.

The radius of the circle is

$$R = \frac{|(p - w)(d - pc)|\sqrt{1 - d^2}}{|F|} .$$

For $F = 0$ and $\Delta \neq 0$, the conic is a parabola.

Using the rotated system, Equation (39) reduces to

$$(d - pc)^2 y'^2 - 2mx'(p - w)(pd - c) + \left(d^2 - c^2\right)(p - w)^2 = 0 \tag{53}$$

or in a standard form,

$$y'^2 = \frac{2m(p - w)(pd - c)}{(d - pc)^2}\left[x' - \frac{\left(d^2 - c^2\right)(p - w)^2}{2m\,(pd - c)}\right] \tag{54}$$

This parabola has a center, $H(x_0', y_0')$, with

$$x_0' = \frac{\left(d^2 - c^2\right)(p - w)}{2m\,(pd - c)} \tag{55}$$

and

$$y_0' = 0 .$$

The coordinates of the focus are

$$\left(x_0' + h,\ 0\right) \tag{56}$$

where

$$h = \frac{m(p - w)(pd - c)}{2(d - pc)^2} \ ; \text{ note: } \sqrt{h^2} = \text{focal length.} \tag{57}$$

The center of the parabola in this rotated system is given by

$$\vec{H}' = x_0' \ \hat{i} + 0 \cdot \hat{j} + w\hat{k} \ .$$

Rotating the system to its original position, as before, we find

$$\vec{H} = R^{-1} \ \vec{H}' = x_0' \ \frac{a}{m} \ \hat{i} + x_0' \ \frac{b}{m} \ \hat{j} + w\hat{k} \ . \tag{58}$$

The cartesian coordinates of the center (in the image plane) are

$$x_H' = \frac{a\left(d^2 - c^2\right)(p - w)}{2m^2 \ (pd - c)}$$

and

$$y_H' = \frac{b\left(d^2 - c^2\right)(p - w)}{2m^2 \ (pd - c)} \ . \tag{59}$$

Using a similar procedure, the coordinates of the focus are

$$x_f' = \frac{a x_0'}{m} - a$$

and

$$y'_f = \frac{b}{m} x'_0 + \frac{ah}{m} . \qquad (60)$$

THE DEGENERATE CASES, $\Delta = 0$

The following cases are considered.

Case 1. $p = w$.

In this study $p \neq w$.

Case 2. $\left(1 - d^2\right) = 0$.

Project a point ellipse and if $c \neq 1$ the image is given by Equation (44).

If $c = +1$, we have no image if $(d - pc) = 0$; otherwise, the image is the point $(0, 0)$.

Case 3. $d - pc = 0$

Using the rotated system, Equation (39) reduces to

$$x'^2 m^2\left(p^2 - 1\right) - 2mx' (p - w)(pd - c) + \left(d^2 - c^2\right)(p - w)^2 = 0 \qquad (61)$$

Solving Equation (61) we have

$$x' = \frac{(p - w)(pd - c)}{m(p^2 - 1)} \quad \text{for } p \neq 1 . \qquad (62)$$

If, in addition $p^2 = 1$, Equation (61) reduces to (for $m \neq 0$)

$$x' = \frac{(d + c)(1 - w)}{2m} \quad \text{for } p = +1 , \qquad (63)$$

and

$$x' = \frac{(d - c)(1 + w)}{2m} \quad \text{for } p = -1 . \tag{64}$$

To find the equation of the image in the original system, let $\vec{d}_0 = d\hat{i} + w\hat{k}$ be a vector normal to the lines given in Equation (62) and Equation (63), where $d \neq 0$ is the directed distance from $(0, 0, w)$ to the line.

Then, rotating d into the original system we have

$$\vec{d} = \frac{Da}{m}\hat{i} + \frac{Db}{m}\hat{j} + w\hat{k}.$$

The projection of d on the x' y' plane is

$$\vec{d'} = \frac{Da}{m}\hat{i} + \frac{Db}{m}\hat{j}$$

then $\vec{x} \cdot \frac{\vec{d}}{D} = D$ gives the equation of the line as

$$\frac{ax'}{m} + \frac{b}{m}y' = D .$$

The images corresponding to Equations (62), (63), and (64) are

for Equation (62)

$$ax' + by' = \frac{(p - w)(pd - c)}{(p - 1)} ,$$

for Equation (63)

$$ax' + by' = \frac{(d + c)(1 - w)}{2} \quad \text{for } p = +1,$$

and for Equation (64)

$$ax' + by' = \frac{(d - c)(1 + w)}{2} \quad \text{for } p = -1$$

Case 4. $c = d = 0.$

In this case the zone boundary and the point of projection lie in the plane $ax + by = 0$. The image obtained is, then, the intersection of this plane with the image plane, $Z = w$. The image is, then, the line

$$ax' + by' = 0 .$$

SECTION VIII

POSITIONING THE IMAGE PLANE

There is no restriction on the analysis if $p \geq -1$.

A point $T(x, y, z)$ of the sphere with position vector $\hat{T} = x\hat{i} + y\hat{j} + z\hat{k}$ has a projection on the (x, y) plane given by

$$\vec{T}_{xy} = x\hat{i} + y\hat{j} \ .$$

The position of the image of point T is

$$\vec{T'} = \frac{p - w}{p - z} \cdot (x\hat{i} + y\hat{j}) \ .$$

Thus, if $(p - w/p - z) > 0$, the image will have the same direction from the origin as the preimage, otherwise the direction will be increased by 180 degrees.

The factor $(p - w/p - z) > 0$ if and only if

$$w > p \text{ when } z > p$$

or

$$w < p \text{ when } z < p \ .$$

Now, if $-1 \leq p < Q_3'$ then all points (x, y, z) of the preimage have $z > p$ hence we require $w > p$.

If $Q_3' < p < Q_3$, then, for some points of the preimage, $z < p$, and only that portion of the preimage for which $z > p$ shall be used. We require, also, that $w > p$.

If $p > 1 > Q_3$, then for all points (x, y, z) of the preimage, $z < p$, and we require then that $w < p$.

If $p > 1$, project only that portion of the preimage for which $z \geq 1/p$. It can be shown that a right circular cone whose apex is at P is tangent to the sphere on the circle given by

$$x^2 + y^2 + z^2 = 1$$

and

$$z = \frac{1}{p} .$$

We shall project those points of the preimage for which $z > 1/p$ and we shall require $w < p$.

APPENDIX

DERIVATION OF THE DISCRIMINANT Δ

The expansion of the determinant giving Δ is a rather tricky algebraic exercise. For this reason the essentials are given below.

Substituting from Equation 15

$$\Delta = 1/2 \begin{vmatrix} 2\left[(d - pc)^2 + a^2(p - 1)\right] & 2ab\left(p^2 - 1\right) & 2a(c - pd)\,(p - w) \\ 2ab\left(p^2 - 1\right) & 2\left[(pc - d)^2 + b^2\left(p^2 - 1\right)\right] & 2b(c - pd)\,(p - w) \\ 2a(c - pd)\,(p - w) & 2b(c - pd)\,(p - w) & 2\left(d^2 - c^2\right)\,(p - w)^2 \end{vmatrix},$$

An obvious factoring gives,

$$\Delta = 4\,(p - w)^2 \begin{vmatrix} (d - pc)^2 + a^2\left(p^2 - 1\right) & ab\left(p^2 - 1\right) & a(c - pd) \\ ab\left(p^2 - 1\right) & (pc - d)^2 + b^2\left(p^2 - 1\right) & b(c - pd) \\ a(c - pd) & b(c - pd) & \left(d^2 - c^2\right) \end{vmatrix}.$$

Factoring out a^2b^2, then subtracting column 2 from column 1, then factoring out $(d - pc)^2$, and subtracting row 1 from row 2, we have

$$\Delta = 4a^2b^2\,(p - w)^2\,(d - pc)^2 \begin{vmatrix} \frac{1}{a^2} & \left(p^2 - 1\right) & c - pd \\ -\left(\frac{1}{a^2} + \frac{1}{b^2}\right) & \frac{(d - pc)^2}{b^2} & 0 \\ 0 & (c - pd) & d^2 - c^2 \end{vmatrix}.$$

Finally, expanding on the first column,

$$\Delta = -4(d - pc)^4 (p - w)^2 \left(1 - d^2\right).$$

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13. ABSTRACT

This report considers the projection of spherical zone boundaries onto an image plane using the "perspective projection". Special cases of this type of mapping includes: the stereographic, the gnomonic, and the orthographic projections.

In addition to proving that the image of any spherical zone boundary is a conic section, the formulation of the parameters of that conic section is given.

An important aspect of the projection technique lies in the capability of having any point of the sphere as the center of projection and any great circle through that point as the centerline of the mapping.

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There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.